INDIAN STATISTICAL INSTITUTE, Bangalore Centre
Midsem
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Course Name: Analysis of several variables

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1. Prove that $GL(n, \mathbb{C})$ is connected in the following suggested way. Suppose $A, B \in GL(n, \mathbb{C})$. First show that there are only a finitely many complex numbers such that $\det(\lambda A + (1-\lambda)B) = 0$. Then using this information show that there exists a path $\rho(t) = \lambda(t)A + (1-\lambda(t))B \subseteq GL(n, \mathbb{C})$ such that $\lambda(0) = 0$ and $\lambda(1) = 1$. [7]

2. Let V be a vector space over $\mathbb{R}$ and let $\|.\| : V \to \mathbb{R}$ satisfying all properties of being a norm except the triangle inequality. Then $\|.\|$ is a norm if and only if the unit ball $\{v \in V : \|v\| \leq 1\}$ is convex. [5]

3. TRUE or FALSE with justification.

   [1] Let $f : \mathbb{R}^n \to \mathbb{R}$ such that $|f(x)| \leq |x|^2$. Then f is differentiable at the origin. [2]

   [2] Let f be a bilinear function on $\mathbb{R}^2$. Then f is differentiable on $\mathbb{R}^2$. [3]

   [3] Let $A = \{(x, y) \in \mathbb{R}^2 : \text{either } x < 0 \text{ or } x \geq 0, y \neq 0\}$. Let $D_2 f \equiv 0$ on A. Then f is not necessarily independent of y on A. [3]

   [4] Let the point $r_0 = (x_0, y_0, z_0)$ lie on the surface $G(x, y, z) = 0$. Assume that $\nabla G(x_0, y_0, z_0) \neq 0$. Suppose that the parametrized curve $r(t) = (x(t), y(t), z(t))$ is contained in the surface and that $r(t_0) = r_0$. Then the tangent line to the curve at r_0 lies in the tangent plane to $G = 0$ at r_0. [2]

   [5] Let $F(u)$ be any differentiable function of one variable. Define $z(x, y) = F(x^2+y^2)$. Then the partial differential equation

   $$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0.$$

   [2]

   [6] The function $f(x, y) = x^2 + y^2$ has a maxima on the curve $xy = 1$. [3]

4. For nonzero real numbers a, b, c, define $F : \mathbb{R}^2 \to \mathbb{R}$ by $F(x, y) = (a/2)y^2 + by + cx$. Then

   [1] Is it true that there exist r_0 and a C^1 function $f : (-r_0, r_0) \to \mathbb{R}$ such that $f(0) = 0$ and $F(x, f(x)) = 0$ for all $x \in (-r_0, r_0)$. [2]

   [2] Use the quadratic formula to answer the same question as above, explicitly finding r_0 and f. In particular, can you find the largest possible r_0.

   [6]